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Sensitivity Analysis Algorithm for the State Chi-Square Test

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In recent years, a powerful failure detection approach, called the state chi-square test, has attracted much attention since it not only avoids assumptions about how the failure components behave but also is applicable to time-varying systems where additive Gaussian white process and measurement noise are present. Owing to unavoidable inaccuracies in the model, robustness relative to modeling error has been recognized as a critical issue in the design of failure detection and isolation systems. A sensitivity analysis algorithm is presented that provides a unified tool for statistical analysis of the impact of modeling errors on the failure detection systems using the state chi-square test. The sensitivity analysis results can be grouped, for example, by using the neural network techniques, into robust decision regions to minimize the impact of modeling errors and to maximize the detection sensitivity.

I. Introduction

AN essential prerequisite for reliability and safety of a control process is early failure detection and isolation (FDI), i.e., the detection of the presence of failure and the isolation of the equipment responsible for the irregularity. During recent years, a variety of FDI approaches have been developed.[1–5] Among them, the parity equations[6–9] and the failure detection filter[8,10–13] are two kinds of well-known model-based FDI approaches that avoid assumptions about how the failed component behaves. Owing to unavoidable inaccuracies in the model, robustness relative to modeling error has been recognized as a critical issue of the FDI system design. Here robustness means the degree to which the failure detection performance is unaffected by (or remains insensitive to) uncertainties in the system. For complex and time-varying systems, modeling errors are almost inevitably present, which may interfere seriously with the FDI procedure and thus offer challenges to the design of a reliable and robust FDI system. Although a number of effective robust FDI approaches have been developed recently,[5,12,14,15] their application is still limited for many reasons. For example, the well-known disturbances (unknown inputs) decoupling approach[5] and the eigenstructure assignment approach[12] require that the disturbance distribution matrix (or disturbance structure) be modeled accurately, which is evidently very difficult for most complex systems. Most of the current research in FDI approaches (as well as their robustness to modeling errors) has been mainly carried out with regard to a benign context, i.e., time-invarying linear structures devoid of noise terms, where exclusive use of observers suffice for the FDI.

There are other powerful failure detection approaches that not only avoid assumptions about how the failure components behave but also are applicable to time-varying systems where additive Gaussian white process and measurement noise are present. These approaches detect failures by monitoring the consistency of the state estimates of the Kalman filter and the state propagator with respect to the two-ellipsoid overlap test[16] or more simply the chi-square test,[17] called the state chi-square test (SCST). Two novel methods were proposed in Ref. 18 to improve the failure sensitivity of the SCST. The first method calls for the use of a pair of state propagators that are alternatively reset with data from the Kalman filter to increase their accuracy and thereby the failure sensitivity of the detection system. The second method is based on monitoring the state estimates of the Kalman filter individually, so that the test statistics are related to the state estimates most seriously affected by the failures. This serves to enhance the failure sensitivity of the detection system.

The decision functions of these SCST approaches[17–20] were derived under the assumption that the mathematical model within both the Kalman filter and the state propagator described completely the real stochastic dynamical system structure. In practice, modeling errors are unavoidable either because dynamic and statistical parameters of a system are not exactly known, or because a reduced-order filter (using just the most significant states) has to be implemented for on-line applications. This is due to practical constraints on available computational resources. Thus, the estimate of the Kalman filter may not be consistent with that of the state propagator not only because of a failure but also because of modeling errors. Although raising the decision threshold to compensate does reduce false alarms, it makes the test less sensitive to actual failures and may result in missed detections when failures do occur.

In this paper, a sensitivity analysis algorithm is presented for the statistical analysis of the SCST performance subject to model errors. Under an assumption that there exists an underlying state variable truth model of fairly high dimension that completely describes the detailed error evolution of the system, the algorithm is developed by investigating the actual mean and covariance matrix of the difference between the Kalman filter and the state propagator. The impact of model errors on the failure decision making can be evaluated by comparing the true statistical distribution of the test statistic to that when there are no model errors. The sensitivity analysis results can be further grouped, for example, by using neural network techniques, into robust decision regions to minimize the impact of modeling errors and to maximize the detection sensitivity.

II. State Chi-Square Test

Assume the model for the design of the Kalman filter is described as follows:

$$x^d(k+1) = \Phi^d(k)x^d(k) + \Gamma^d(k)w^d(k) \tag{1}$$

$$y(k) = H^d(k)x^d(k) + v^d(k) \tag{2}$$

$$E[w^d(k)] = 0 \qquad E[w^d(k)w^{dT}(j)] = Q^d(k)\delta_{kj} \tag{3}$$

$$E[v^d(k)] = 0 \qquad E[v^d(k)v^{dT}(j)] = R^d(k)\delta_{kj} \tag{4}$$

where the superscript d stands for the design model, the superscript t stands for the truth model, and $x^d(k)$ is an n-dimensional state, with Gaussian initial condition $x^d(0)$, which has mean x_0^d and covar P_0; $x^d(0)$, $w^d(k)$, and $v^d(k)$ are assumed statistically independent; $Q^d(k)$ is positive semidefinite, and $R^d(k)$ is positive definite; δ_{ij} is

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Kroneker delta (this is 1 when $i = j$ and 0 otherwise). Superscript T represents the transpose operation on a matrix.

Assume that the standard Kalman filter, which gives the state estimate $\hat{x}_K(k \mid k)$ of $x^d(k)$ based upon this design model, is specified by the following equations (the subscript K denotes the Kalman filter).

Time update:

$$\hat{x}_K(k+1 \mid k) = \Phi^d(k)\hat{x}_K(k \mid k) \tag{5}$$

$$P_K(k+1 \mid k) = \Phi^d(k)P_K(k \mid k)\Phi^{dT}(k) + \Gamma^d(k)Q^d(k)\Gamma^{dT}(k) \tag{6}$$

Measurement update:

$$\hat{x}_K(k+1 \mid k+1) = \hat{x}_K(k+1 \mid k) + K_k(k+1)\big[y(k+1) - H^d(k+1)\hat{x}_K(k+1 \mid k)\big] \tag{7}$$

$$K_k(k+1) = P_K(k+1)H^{dT}(k+1)\big[H^d(k+1) \times P_K(k+1)H^{dT}(k+1) + R^d(k+1)\big] \tag{8}$$

$$P_K(k+1 \mid k+1) = \big[I - K_k(k+1)H^d(k+1)\big] \times P_K(k+1 \mid k)\big[I - K_k(k+1)H^d(k+1)\big]^T + K_k(k+1)R^d(k+1)K_k^T(k+1) \tag{9}$$

Initial conditions for the algorithm are

$$\hat{x}_K(0 \mid 0) = x_0^d \qquad P_K(0 \mid 0) = P_0 \tag{10}$$

The SCST detects failure by monitoring the state estimate of the Kalman filter with the chi-square test.[17] A state propagator is usually used as a reference system, whose state estimate $\hat{x}_S(k)$ and covariance matrix $P_S(k)$ (the subscript S denotes the state propagator) are propagated on the sole basis of the system model:

$$\hat{x}_S(k+1) = \Phi^d(k)\hat{x}_S(k) \tag{11}$$

$$P_S(k+1) = \Phi^d(k)P_S(k)\Phi^T(k) + \Gamma^d(k)Q^d(k)\Gamma^{dT}(k) \tag{12}$$

with the initial conditions

$$\hat{x}_S(0) = x_0^d \qquad P_S(0) = P_0 \tag{13}$$

If the design model of Eqs. (1–4) represents completely the real physical system at hand, the difference vector

$$b(k) \triangleq \hat{x}_K(k) - \hat{x}_S(k \mid k) \tag{14}$$

is a Gaussian random vector with zero mean and covariance $B(k)$[17]

$$B(k) \triangleq E[b(k)b^T(k)] = P_S(k) - P_K(k \mid k) \tag{15}$$

Thus, the random variate $\zeta(k)$

$$\zeta(k) \triangleq b^T(k)B^{-1}(k)b(k) \tag{16}$$

is a chi-square distributed random variable with n degrees of freedom, i.e., $\zeta(k) \sim \chi^2(n)$. Hence, the following failure detection rule can be considered

$$\begin{aligned} &\text{if} \quad \zeta(k) \geq \epsilon \quad \text{then there is a failure} \\ &\text{if} \quad \zeta(k) < \epsilon \quad \text{then there is no failure} \end{aligned} \tag{17}$$

where ϵ is a chosen threshold.

Note that the preceding failure detection rule is derived under the assumption that the mathematical model for both the Kalman filter and the state propagator describes completely the real stochastic dynamical system structure. In practice, however, modeling errors are unavoidable either because dynamic and statistical parameters of a system are not exactly known or because a reduced-order filter (using just the most significant states) has to be implemented for on-line applications, due to the practical constraints on available computational resources. Then $b(k)$ is usually a biased Gaussian vector and its covariance cannot be simply calculated as in Eq. (15). Thus, the $\zeta(k)$ is actually not a chi-square distributed variable. Although raising the decision threshold to compensate does reduce false alarms, it makes the test less sensitive to actual failures and may result in missed failure detections. It is important to carefully investigate the sensitivity of the statistics of ζ before the detection rule is applied to real applications.

III. Truth Statistics of $b(k)$

Before analyzing the truth statistics of $\zeta(k)$, the true statistical behavior of $b(k)$ needs to be determined. Assume the truth model for the system is represented as follows:

$$x^t(k+1) = \Phi^t(k)x^t(k) + \Gamma^t(k)w^t(k) \tag{18}$$

$$y(k) = H^t(k)x^t(k) + v^t(k) \tag{19}$$

$$E[w^t(k)] = 0 \qquad E[w^t(k)w^{tT}(j)] = Q^t(k)\delta_{kj} \tag{20}$$

$$E[v^t(k)] = 0 \qquad E[v^t(k)v^{tT}(j)] = R^t(k)\delta_{kj} \tag{21}$$

where $x^t(k)$ is an $\bar{n}$-dimensional state ($\bar{n} \geq n$), with Gaussian initial condition $x^t(0)$, which has mean x_0^t and covariance P_0^t. $x^t(0)$, $w^t(k)$, and $v^t(k)$ are assumed to be mutually statistically independent; $Q^t(k)$ is positive semidefinite, and $R^t(k)$ is positive definite.

Then, the truth estimation errors of the Kalman filter and the state propagator can be defined as follows:

$$\tilde{x}_K^t(k \mid k) \triangleq Tx^t(k) - \hat{x}_K(k \mid k) \tag{22}$$

$$\tilde{x}_S^t(k) \triangleq Tx^t(k) - \hat{x}_S(k) \tag{23}$$

where T is an $n \times \bar{n}$ selector matrix that frequently is of the following simple form:

$$T = \{I(n \times n)\ 0[n \times (\bar{n} - n)]\} \tag{24}$$

with I being a unity matrix. Then, the estimation difference $b(k)$ is expressed as:

$$b(k) = \hat{x}_K(k \mid k) - \hat{x}_S(k) = \tilde{x}_S^t(k) - \tilde{x}_K^t(k \mid k) \tag{25}$$

Equation (25) shows that the true statistical behavior of $b(k)$ depends on the relationship between $\tilde{x}_S^t(k)$ and $\tilde{x}_K^t(k \mid k)$, whose propagation equations are derived next.

From Eqs. (22), (7), (18), and (19), we have

$$\begin{aligned} \tilde{x}_K^t(k+1 \mid k+1) &= Tx^t(k+1) - \hat{x}_K(k+1 \mid k+1) \\ &= Tx^t(k+1) - \big[I - K_k(k+1)H^d(k+1)\big]\hat{x}_K(k+1 \mid k) \\ &\quad - K_k(k+1)y(k+1) = \big[T - K_k(k+1)H^t(k+1)\big] \\ &\quad \times x^t(k+1) - \big[I - K_k(k+1)H^d(k+1)\big]\Phi^d(k)\hat{x}_K(k \mid k) \\ &\quad - K_k(k+1)v^t(k+1) = \big[T - K_k(k+1)H^t(k+1)\big]\Phi^t x^t(k) \\ &\quad - \big[I - K_k(k+1)H^d(k+1)\big]\Phi^d(k)\big[Tx^t(k) - \tilde{x}_K^t(k \mid k)\big] \\ &\quad + \big[T - K_k(k+1)H^t(k+1)\big]\Gamma^t(k)w^t(k) \\ &\quad - K_k(k+1)v^t(k+1) = \big[I - K_k(k+1)H^d(k+1)\big] \\ &\quad \times \Phi^d(k)\tilde{x}_K^t(k \mid k) + \big\{\big[T\Phi^t(k) - \Phi^d(k)T\big] \\ &\quad - K_k(k+1)[H^t(k+1)\Phi^t(k) - H^d(k)\Phi^d(k)T]\big\}x^t(k) \\ &\quad + \big[T - K_k(k+1)H^t(k+1)\big]\Gamma^t(k)w^t(k) \\ &\quad - K_k(k+1)v^t(k+1) \end{aligned} \tag{26}$$

The propagation equation of $\tilde{x}_S^t(k)$ is obtained from Eqs. (23), (18), and (11):

$$\begin{aligned}\tilde{x}_S^t(k+1) &= Tx^t(k+1) - \hat{x}_S(k+1)\\ &= T\Phi^t(k)x^t(k) + T\Gamma^t(k)w^t(k) - \Phi^d(k)\hat{x}_S(k)\\ &= \Phi^d(k)\tilde{x}_S^t(k) + [T\Phi^t(k) - \Phi^d(k)T]x^t(k) + T\Gamma^t(k)w^t(k)\end{aligned} \tag{27}$$

Now we can define an augmented state vector process $z(k)$ and an augmented noise vector process $u(k)$ as

$$z(k) = \begin{bmatrix} z_1(k)\\ z_2(k)\\ z_3(k)\end{bmatrix} = \begin{bmatrix} \tilde{x}_K^t(k \mid k)\\ \tilde{x}_S^t(k)\\ x^t(k)\end{bmatrix} \tag{28}$$

$$u(k) = \begin{bmatrix} w^t(k)\\ v^t(k+1)\end{bmatrix} \tag{29}$$

From Eqs. (18), (25), and (27), the propagation equation of $z(k)$ is known to be

$$z(k+1) = A(k)z(k) + C(k)u(k) \tag{30}$$

where

$$A(k) = \begin{bmatrix} A_{11} & 0 & A_{13}\\ 0 & A_{22} & A_{23}\\ 0 & 0 & A_{33}\end{bmatrix} \tag{31}$$

$$C(k) = \begin{bmatrix} C_{11} & C_{12}\\ C_{21} & 0\\ C_{31} & 0\end{bmatrix} \tag{32}$$

with

$$A_{11} = \left[I - K_k(k+1)H^d(k+1)\right]\Phi^d(k) \tag{33}$$

$$\begin{aligned}A_{13} &= T\Phi^t(k) - \Phi^d(k)T - K_k(k+1)\\ &\quad\times [H^t(k+1)\Phi^t(k) - H^d(k)\Phi^d(k)T]\end{aligned} \tag{34}$$

$$A_{22} = \Phi^d(k) \tag{35}$$

$$A_{23} = [T\Phi^t(k) - \Phi^d(k)T] \tag{36}$$

$$A_{33} = \Phi^t(k) \tag{37}$$

$$C_{11} = \left[T - K_k(k+1)H^t(k+1)\right]\Gamma^t(k) \tag{38}$$

$$C_{12} = -K_k(k+1) \tag{39}$$

$$C_{21} = T\Gamma^t(k) \tag{40}$$

$$C_{31} = \Gamma^t(k) \tag{41}$$

The expected value of $z(k)$, denoted by $\bar{z}(k)$, and its covariance, denoted by $Z(k)$, are readily obtained by the propagation relations

$$\bar{z}(k+1) \triangleq E[z(k+1)] = A(k)\bar{z}(k) \tag{42}$$

$$\begin{aligned}Z(k+1) &\triangleq E\{[z(k+1) - \bar{z}(k+1)][z(k+1) - \bar{z}(k+1)]^T\}\\ &= A(k)Z(k)A^T(k) + C(k)D(k)C^T(k)\end{aligned} \tag{43}$$

where

$$D(k) \triangleq E[u(k)u^T(k)] = \text{diag block}\,[Q^t(k)R^t(k+1)] \tag{44}$$

The initial conditions for the propagation of $\bar{z}$ and Z are

$$\bar{z}(0) = E\begin{bmatrix} \tilde{x}_K^t(0)\\ \tilde{x}_S^t(0)\\ x^t(0)\end{bmatrix} = \begin{bmatrix} Tx_0^t - x_0^d\\ Tx_0^t - x_0^d\\ x_0^t\end{bmatrix} \tag{45}$$

and

$$\begin{aligned}Z(0) &= E\{[z(0) - \bar{z}(0)][z(0) - \bar{z}(0)]^T\}\\ &= \begin{bmatrix} Z_{11}(0) & Z_{12}(0) & Z_{13}(0)\\ Z_{21}(0) & Z_{22}(0) & Z_{23}(0)\\ Z_{31}(0) & Z_{32}(0) & Z_{33}(0)\end{bmatrix}\end{aligned} \tag{46}$$

where

$$\begin{aligned}Z_{11}(0) &= Z_{22}(0) = Z_{12}(0) = Z_{21}(0)\\ &= E\Big\{\Big[\big(Tx^t(0) - x_0^d\big) - \big(Tx_0^t - x_0^d\big)\Big]\\ &\quad\times \Big[\big(Tx^t(0) - x_0^d\big) - \big(Tx_0^t - x_0^d\big)\Big]^T\Big\} = TP_0^tT^T\end{aligned} \tag{47}$$

and

$$\begin{aligned}Z_{13}(0) &= Z_{23}(0) = Z_{31}^T(0) = Z_{32}^T(0)\\ &= E\Big\{\Big[\big(Tx^t(0) - x_0\big) - \big(Tx_0^t - x_0\big)\Big]\Big[x^t(0) - x_0^t\Big]^T\Big\} = TP_0^t\end{aligned} \tag{48}$$

$$Z_{33}(0) = P_0^t \tag{49}$$

After the $\bar{z}$ and Z are solved by Eqs. (42) and (43), the truth mean and covariance of $b(k)$ are readily obtained by

$$\bar{b}(k) \triangleq E\left[\tilde{x}_S^t(k) - \tilde{x}_K^t(k \mid k)\right] = \bar{z}_2(k) - \bar{z}_1(k) \tag{50}$$

and

$$\begin{aligned}B^t(k) &\triangleq E\{[b(k) - \bar{b}(k)][b(k) - \bar{b}(k)]^T\}\\ &= E\Big(\{[(z_2(k) - z_1(k)] - [\bar{z}_2(k) - \bar{z}_1(k)]\}\\ &\quad\times \big\{[z_2(k) - z_1(k)] - [\bar{z}_2(k) - \bar{z}_1(k)]^T\big\}\Big)\\ &= Z_{11}(k) + Z_{22}(k) - Z_{12}(k) - Z_{21}(k)\end{aligned} \tag{51}$$

IV. Truth Distribution of $\zeta(k)$

Because of model error effects, we have

$$E[b(k)] = \bar{b}(k) \neq 0 \tag{52}$$

$$\begin{aligned}B^t(k) &= E\{[b(k) - \bar{b}(k)][b(k) - \bar{b}(k)]^T\}\\ &\neq P_S(k) - P_K(k \mid k) = B(k)\end{aligned} \tag{53}$$

The random variate $\zeta(k)$ defined in Eq. (16) is clearly no longer a chi-square distributed variable. But once we have the truth mean and covariance of $b(k)$, the truth distribution of the random variate $\zeta(k)$ can be determined, as is shown in the following. Before we discuss the method for calculating the truth distribution function of $\zeta(k)$, let us look into its interesting structure. Since $B^t(k)$ is positive definite, there exists a real nonsingular matrix G such that

$$B^t(k) = G(k)G^T(k) \tag{54}$$

Let

$$D(k) = G^T(k)B^{-1}(k)G(k) \tag{55}$$

then $D(k)$ is symmetric and hence there exists an orthogonal matrix $\mathcal{O}(k)$ such that

$$\mathcal{O}^T(k)D(k)\mathcal{O}(k) = \mathcal{O}^T(k)G^T(k)B^{-1}(k)G(k)\mathcal{O}(k) = \Lambda(k) \tag{56}$$

where $\Lambda(k)$ is a diagonal matrix with real diagonal elements $\lambda_1(k), \ldots, \lambda_n(k)$, which are the eigenvalues of the matrix $D(k)$. Equivalently, they are the eigenvalues of the matrix $B^{-1}(k)B^t(k)$. Considering a transformation $b(k) = M(k)d(k)$, where $M(k) =$

$G(k)\mathcal{O}(k)$, the elements of $\boldsymbol{d}(k)$ are nonzero mean and independent Gaussian variables since

$$E[\boldsymbol{d}(k)] = \bar{\boldsymbol{d}}(k) = M^{-1}(k)\bar{\boldsymbol{b}}(k) \tag{57}$$

$$\begin{aligned} E\{[\boldsymbol{d}(k) - \bar{\boldsymbol{d}}(k)][\boldsymbol{d}(k) - \bar{\boldsymbol{d}}(k)^T]\} &= M^{-1}(k)E\{[\boldsymbol{b}(k) - \bar{\boldsymbol{b}}(k)] \\ \times [\boldsymbol{b}(k) - \bar{\boldsymbol{b}}(k)]^T\}M^{-T}(k) &= M^{-1}(k)B^t(k)M^{-T}(k) \\ = \mathcal{O}^{-1}(k)G^{-1}(k)G(k)G^T(k)G^{-T}(k)P^{-T}(k) &= I \end{aligned} \tag{58}$$

From

$$\begin{aligned} \zeta(k) &= \boldsymbol{b}^T(k)B^{-1}(k)\boldsymbol{b}(k) = \boldsymbol{d}^T(k)M^T(k)B^{-1}(k)M(k)\boldsymbol{d}(k) \\ &= \boldsymbol{d}^T(k)\mathcal{O}^T(k)G^T(k)B(k)G(k)\mathcal{O}(k)\boldsymbol{d}(k) \\ &= \boldsymbol{d}^T(k)\Lambda(k)\boldsymbol{d}(k) = \sum_{i=1}^{n}\lambda_i(k)d_i^2(k) \end{aligned} \tag{59}$$

it is known that $\zeta(k)$ is actually a weighted sum of n independent noncentral chi-square distributed variables $d_i^2(k)$, since $d_i^2(k) \sim \chi^2[\bar{d}_i(k), 1]$.

Calculation of the distribution of $\zeta(k)$ might be conducted in several ways. One of the obvious ways is to calculate the distribution function of $\zeta(k)$ in terms of its characteristic function. As $\boldsymbol{b}(k) \sim N[\bar{\boldsymbol{b}}(k), B^t(k)]$, $[B^t(k) > 0]$, the characteristic function $\phi(t)$ of $\zeta(k)$ was given as[21]

$$\begin{aligned} \phi(t) &\triangleq E[e^{it\zeta}] = |I - 2itB^{-1}B^t|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\bar{\boldsymbol{b}}^T B^{-r}\bar{\boldsymbol{b}}\right. \\ &\left.+ \tfrac{1}{2}\bar{\boldsymbol{b}}^T B^{-r}(B^{-r} - 2itB^{-1})B^{-r}\bar{\boldsymbol{b}}\right\} \end{aligned} \tag{60}$$

Then the density function $g(\zeta)$ of ζ might be obtained from the general inversion rule

$$g(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\zeta}\phi(t)\,\mathrm{d}t \tag{61}$$

and the distribution function $F(\zeta)$ is obtained by integration of the density function $g(\zeta)$. It is noted, however, that the integration of Eq. (61) is usually too complex to be solved directly when $\bar{b}(k) \neq 0$.

A more convenient way to get directly the distribution function of $F(\zeta)$ is to use the Gil–Pelaez inversion formular,[22] where $F(\zeta)$ is expressed as

$$F(\zeta < c) = \frac{1}{2} - \int_{-\infty}^{\infty} \mathrm{Im}\left[\frac{\phi(t)e^{-itc}}{2\pi t}\right]\mathrm{d}t \tag{62}$$

Starting from this formula, an algorithm for numerical calculation of the distribution function $F(\zeta)$ is given in Ref. 22. The distribution function $F(\zeta)$ can also be calculated because $F(\zeta)$ can be presented as an infinite series centralized chi-square distribution function, as indicated by Kotz et al.[23]

After the truth distribution of the test statistic $\zeta(k)$ of the state chi-square test is calculated, the impact of model errors on the FDI decision making can be determined by comparing the true statistical distribution of $\zeta(k)$ with the one determined by assuming that there are no modeling errors.

V. Conclusions

In this paper, a sensitivity analysis algorithm was developed that provides a unified tool for statistical analysis of the impact of modeling errors on the performance of FDI systems that use the state chi-square test. Under the assumption that there exists an underlying state variable truth model of fairly high dimension that completely describes the detailed error evolution of the system, the sensitivity analysis algorithm can be used to calculate the actual mean and covariance matrix of the state estimation difference between the Kalman filter and the state propagator. It can also find the truth distribution of the statistics of the state chi-square test. Then the impact of modeling errors on failure decision can be ascertained by comparing the true statistical distribution of the test statistics with the one with no modeling errors. The sensitivity analysis results can be further grouped, by using intelligent data classification techniques into robust FDI decision regions. This minimizes the impact of modeling errors and maximizes the detection sensitivity. The sensitivity analysis algorithm is currently being applied to analyze the effects of modeling errors on Global Positioning System integrity monitoring systems.[24,25]

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